Resit Exam — Analysis (WBMA012-05)

Thursday 11 April 2024, 8.30h–10.30h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. Provide clear arguments for all your answers: only answering "yes", "no", or "42" is not sufficient. You may use all theorems and statements in the book, but you should clearly indicate which of them you are using.
3. The total score for all questions equals 90. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (5 + 10 = 15 points)

Assume that $A \subseteq \mathbb{R}$ is nonempty (but not necessarily bounded) and define the function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \sup\{2 - |x - a| \,:\, a \in A\}.$$

(a) Explain why the supremum above exists for all $x \in \mathbb{R}$.

(b) Let $A = (-1, \infty)$ and prove that $f(-3) = 0$.

Problem 2 (8 + 7 = 15 points)

Consider the following sequence:

$$x_{n+1} = \sqrt{12 + x_n} \quad \text{with} \quad x_1 = 0.$$

(a) Show that $x_{n+1} > x_n$ and $x_n < 4$ for all $n \in \mathbb{N}$.

(b) Prove that the sequence (x_n) converges and compute $\lim x_n$.

Problem 3 (5 + 5 + 5 = 15 points)

Consider the set $A = \left\{ \frac{1}{p} - \frac{1}{q} \,:\, p, q \in \mathbb{N} \right\}$.

(a) Find a limit point of A which is contained in A.

(b) Find a limit point of A which is *not* contained in A.

(c) Is the set A compact?

Please turn over for problems 4, 5 and 6!

Problem 4 (15 points)

Consider the following function:

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \sqrt{3 + 4x^2}.$$

Use the Mean Value Theorem to prove that f is uniformly continuous on $\mathbb{R}$.

Problem 5 (3 + 6 + 6 = 15 points)

Assume $f : \mathbb{R} \to \mathbb{R}$ is continuous, and consider the functions

$$f_n : \mathbb{R} \to \mathbb{R}, \quad f_n(x) = f(x + 1/n).$$

Prove the following statements:

(a) The sequence (f_n) converges pointwise to f.

(b) If $f(x) = |x|$, then (f_n) converges uniformly on $\mathbb{R}$.

(c) If $f(x) = x^2$, then (f_n) does *not* converge uniformly on $\mathbb{R}$.

Problem 6 (8 + 7 = 15 points)

Consider the function $f : [0, 1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} -1 & \text{if } x = 0, \\ 1 & \text{if } 0 < x < 1 \\ 2 & \text{if } x = 1. \end{cases}$$

(a) Let $P = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ be any partition of $[0, 1]$. Show that

$$U(f, P) - L(f, P) = 1 + 2x_1 - x_{n-1}.$$

(b) Use part (a) to prove that f is integrable on $[0, 1]$.

End of test (90 points)

Solution of problem 1 (5 + 10 = 15 points)

(a) For a fixed $x \in \mathbb{R}$ we define the set $B_x = \{2 - |x - a| \; : \; a \in A\}$. Since A is nonempty, so is the set B_x.
(1 point)

Note that $2 - |x - a| \leq 2$ for all $a \in A$, which implies that $u = 2$ is an upper bound for the set B_x.
(2 points)

The Axiom of Completeness states that every nonempty set that has an upper bound also has a least upper bound.
(2 points)

(b) Consider the set $B = \{2 - |-3 - a| \; : \; a \in A\}$. We first claim that $u = 0$ is an upper bound for B. This can be done in (at least) two different ways.

Method 1. We have the following implications

$$\begin{aligned} a \in A = (-1, \infty) \quad &\Rightarrow \quad a > -1 \\ &\Rightarrow \quad -a < 1 \\ &\Rightarrow \quad -3 - a < -2 \\ &\Rightarrow \quad |-3 - a| > 2 \\ &\Rightarrow \quad 2 - |-3 - a| < 0, \end{aligned}$$

which shows the claim.
(4 points)

Method 2. By sketching a picture of the real line, we see that the distance between the point $x = -3$ and a point $a \in A$ is larger than 2. In other words: $|-3 - a| > 2$, and thus $2 - |-3 - a| < 0$, for all $a \in A$.
(4 points)

In order to prove that $\sup B = 0$, we can again follow (at least) two different approaches.

Method 1. Let $u \in \mathbb{R}$ be any upper bound for B, so that

$$2 - |-3 - a| \leq u \quad \text{for all} \quad a \in A.$$

(2 points)

Taking the sequence $a_n = -1 + 1/n \in A$ gives

$$2 - |-3 - (-1 + 1/n)| = 2 - |-2 - 1/n| = 1/n \leq u \quad \text{for all} \quad n \in \mathbb{N}.$$

(2 points)

By taking $n \to \infty$ the Order Limit Theorem implies that $0 \leq u$. By definition, it follows that $\sup B = 0$.
(2 points)

Method 2. Let $\epsilon > 0$ be arbitrary and take $a = -1 + \epsilon/2 \in A$.
(2 points)

A straightforward computation shows that

$$2 - |-3 - a| = -\epsilon/2 > 0 - \epsilon.$$

(2 points)

This means that for all $\epsilon > 0$ we have shown that there exists a $b \in B$ (namely, $b = 2 - |-3 - a|$ with $a = -1 + \epsilon/2 \in A$) such that $0 - \epsilon < b$. By characterization of supremum we have that $\sup B = 0$, as desired.
(2 points)

Solution of problem 2 (8 + 7 = 15 points)

(a) Clearly, we have $x_2 = \sqrt{12 + x_1} = \sqrt{12} > 0 = x_1$.
(1 point)

Assume that $x_{n+1} > x_n$ for some $n \in \mathbb{N}$. Then $12 + x_{n+1} > 12 + x_n$. Since the square root is an increasing function, we obtain $x_{n+2} = \sqrt{12 + x_{n+1}} > \sqrt{12 + x_n} = x_{n+1}$. By induction, it follows that $x_n < 4$ for all $n \in \mathbb{N}$.
(3 points)

Clearly, we have $x_1 < 4$.
(1 point)

Assume that $x_n < 4$ for some $n \in \mathbb{N}$. This gives $12 + x_n < 16$. Since the square root is an increasing function, we obtain $x_{n+1} = \sqrt{12 + x_n} < 4$. By induction, it follows that $x_n < 4$ for all $n \in \mathbb{N}$.
(3 points)

(b) Since (x_n) is an increasing sequence which is bounded from above, it follows by the Monotone Convergence Theorem that $\lim x_n$ exists.
(2 points)

Write $x = \lim x_n$. Since the square root is continuous it follows that $x = \sqrt{12 + x}$, which implies that $x^2 - x - 12 = 0$.
(2 points)

The solutions of this quadratic equation are $x = 4$ and $x = -3$.
(2 points)

Since $x_1 = 0$ and the sequence (x_n) increases, we can rule out $x = -3$. We conclude that $\lim x_n = 4$.
(1 point)

Solution of problem 3 (5 + 5 + 5 = 15 points)

(a) Note that with $p = q$ it follows that $0 \in A$.
(1 point)

Next we claim that $x = 0$ is indeed a limit point of A. To that end, consider the sequence $x_n = 1/(2n)$. With $p = n$ and $q = 2n$ it follows that $x_n \in A$ for all $n \in \mathbb{N}$. Clearly, $x_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim x_n = 0$. This shows that $x = 0$ is indeed a limit point of A.
(4 points)

(b) We claim that $x = 1$ is a limit point of A. Indeed, for $p = 1$ and $q = n$ it follows that $x_n = 1 - 1/n \in A$ for all $n \in \mathbb{N}$. Clearly, $x_n \neq 1$ for all $n \in \mathbb{N}$ and $\lim x_n = 1$. This shows that $x = 1$ is indeed a limit point of A.
(4 points)

However, $1 \notin A$ since

$$\frac{1}{p} - \frac{1}{q} \leq 1 - \frac{1}{q} < 1$$

for all $p, q \in \mathbb{N}$.
(1 point)

(c) From part (b) it follows that A does not contain all its limit points. Therefore, A is not closed.
(2 points)

A set $A \subseteq \mathbb{R}$ is compact if and only if A is both closed and bounded. Since the given set A is not closed, it cannot be compact.
(3 points)

Solution of problem 4 (15 points)

Pick $x, y \in \mathbb{R}$ such that $x \neq y$. By the Mean Value Theorem there exists a point c between x and y such that

$$f(x) - f(y) = f'(c)(x-y) = \frac{4c}{\sqrt{3+4c^2}}(x-y).$$

(4 points)

Since $4c^2 < 3 + 4c^2$ it follows that $2|c| < \sqrt{3+4c^2}$ and thus

$$\left|\frac{4c}{\sqrt{3+4c^2}}\right| < 2.$$

(4 points)

Therefore, we obtain

$$|f(x) - f(y)| = \left|\frac{4c}{\sqrt{3+4c^2}}\right| |x-y| < 2|x-y|.$$

By replacing the "$<$"-sign by a "$\leq$"-sign, the inequality is true for all $x, y \in \mathbb{R}$.
(2 points)

Let $\epsilon > 0$ be arbitrary and take $\delta = \epsilon/2$. Then we have

$$|x-y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| \leq 2|x-y| < 2\delta = \epsilon \quad \text{for all} \quad x, y \in \mathbb{R},$$

which shows that f is uniformly continuous on $\mathbb{R}$.
(5 points)

Solution of problem 5 (3 + 6 + 6 = 15 points)

(a) *Method 1.* For fixed $x \in \mathbb{R}$ consider the sequence $x_n = x + 1/n$. Since $\lim x_n = x$ the continuity of f gives

$$\lim f_n(x) = \lim f(x + 1/n) = \lim f(x_n) = f(x).$$

(3 points)

Method 2. Fix a point $x \in \mathbb{R}$. The function f is continuous at x so for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \epsilon.$$

By the Archimedean Property there exists $N \in \mathbb{N}$ such that $N > 1/\delta$. This gives

$$n \geq N \quad \Rightarrow \quad \left| x - \left(x + \frac{1}{n} \right) \right| = \frac{1}{n} \leq \frac{1}{N} < \delta \quad \Rightarrow \quad |f(x) - f(x + 1/n)| < \epsilon.$$

In summary, we have for all $\epsilon > 0$ the existence of $N \in \mathbb{N}$ such that

$$n \geq N \quad \Rightarrow \quad |f(x) - f_n(x)| < \epsilon,$$

which means that f_n converges pointwise to f. (Note that δ, and thus N, depends on the chosen point x, but this is not made explicit in the notation.)
(3 points)

(b) Let $x \in \mathbb{R}$ be arbitrary. The reverse triangle inequality gives

$$\begin{aligned} \big|f_n(x) - f(x)\big| &= \big|f(x + 1/n) - f(x)\big| \\ &= \big|\, |x + 1/n| - |x| \,\big| \\ &\leq \big|x + 1/n - x\big| \\ &= 1/n. \end{aligned}$$

(3 points)

There are (at least) two arguments to show that $f_n \to f$ uniformlyon $\mathbb{R}$.

Method 1. From the above inequality we obtain

$$\sup_{x \in \mathbb{R}} |f_n(x) - f(x)| \leq \frac{1}{n}.$$

Therefore, it follows that

$$\lim \left(\sup_{x \in \mathbb{R}} |f_n(x) - f(x)| \right) = 0.$$

A theorem proven in the lectures implies that $f_n \to f$ uniformly on $\mathbb{R}$.
(3 points)

Method 2. Let $\epsilon > 0$ be arbitrary. By the Archimedean Property there exists $N \in \mathbb{N}$ such that $1/N < \epsilon$. This gives

$$n \geq N \quad \Rightarrow \quad |f(x) - f_n(x)| \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon \quad \text{for all} \quad x \in \mathbb{R},$$

which is the definition of $f_n \to f$ uniformly on $\mathbb{R}$ (the N only depends on ϵ but not on the point x).
(3 points)

(c) If $f(x) = x^2$, then we have

$$|f_n(x) - f(x)| = \left|\frac{2x}{n} + \frac{1}{n^2}\right|.$$

(3 points)

There are (at least) two different arguments to show that the convergence is not uniform on $\mathbb{R}$.

Method 1. Note that for fixed n the function $|f_n(x) - f(x)|$ is unbounded on $\mathbb{R}$, so

$$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \infty.$$

In particular, it is not true that

$$\lim \left(\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| \right) = 0,$$

which means that the convergence is not uniform on $\mathbb{R}$.
(3 points)

Method 2. It suffices to show that the convergence is not uniform on $[0, \infty)$. Indeed, if we want to have

$$n \geq N \quad \Rightarrow \quad |f_n(x) - f(x)| = \frac{2x}{n} + \frac{1}{n^2} < \epsilon,$$

then it is clear that for fixed ϵ this implication cannot be satisfied for all $x \geq 0$. Indeed, if the inequality holds for some $x > 0$, then by repeatedly doubling x we cannot maintain the inequality without also increasing n (and thus N). Therefore, the N must be dependent on x as well.
(3 points)

Solution of problem 6 (8 + 7 = 15 points)

(a) The upper sum is defined by

$$U(f,P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) \quad \text{where} \quad M_k = \sup\{f(x) \,:\, x \in [x_{k-1}, x_k]\}.$$

The lower sum has an analogous expression which is obtained by replacing the supremum by the infimum. For the given function we have

$$\begin{aligned} U(f,P) &= (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_{n-1} - x_{n-2}) + 2(x_n - x_{n-1}), \\ L(f,P) &= -(x_1 - x_0) + (x_2 - x_1) + \cdots + (x_{n-1} - x_{n-2}) + (x_n - x_{n-1}). \end{aligned}$$

(4 points)

Subtracting these expressions gives

$$U(f,P) - L(f,P) = 2(x_1 - x_0) + (x_n - x_{n-1}).$$

(2 points)

Using that $x_0 = 0$ and $x_n = 1$ gives

$$U(f,P) - L(f,P) = 2x_1 + (1 - x_{n-1}) = 1 + 2x_1 - x_{n-1}.$$

(2 points)

(b) *Method 1.* Take an equispaced partition P so that $x_k = k/n$ for all $k = 0, \ldots, n$. The expression obtained in part (a) gives

$$U(f,P) - L(f,P) = 1 + \frac{2}{n} - \frac{n-1}{n} = \frac{1}{n}.$$

(4 points)

Let $\epsilon > 0$ be arbitrary. By the Archimedean Principle there exists $n \in \mathbb{N}$ such that $1/n < \epsilon$, which gives

$$U(f,P) - L(f,P) < \epsilon,$$

which shows that f is integrable on $[0,1]$.
(3 points)

Method 2. Actually, we can also take a partition P with $n = 3$ intervals. For a given $\epsilon > 0$ we simply take $x_1 = \epsilon/4$ and $x_2 = 1 - \epsilon/2$. The expression obtained in part (a) gives

$$U(f,P) - L(f,P) = 1 + \frac{\epsilon}{2} - 1 + \frac{\epsilon}{2} = \epsilon.$$

(7 points)

In all fairness, the argument of method 2 does not work if $\epsilon > 2$. In this case, we can just take any points $0 < x_1 < x_2 < 1$ since

$$\begin{aligned} U(f,P) - L(f,P) &= 2x_1 + (1 - x_2) \\ &= 1 + 2x_1 - x_2 \\ &< 1 + 2x_2 - x_2 \\ &= 1 - x_2 < 2 < \epsilon. \end{aligned}$$

(Not taken into account in grading)